Algebraic Number Theory

Exercise Sheet 1

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Exercise 1. Show that $\mathbb{Z}[X]$ is not a principal ideal domain.
Hint: Consider the ideal $(2, X)$, and show that it is not principal.

Exercise 2. Let $\mathbb{Q} \subset L$ be a number field and α an element in L. Let $P(X)$ be the minimal polynomial of α over $\mathbb{Q}$. Show that α is integral over $\mathbb{Z}$ if and only if $P(X) \in \mathbb{Z}[X]$.
Hint: one may use Gauss Lemma from the “Algebra” lecture.

Exercise 3. Show that the ring $\mathbb{Z}[X]/(X^2 - 3)$ is integral over $\mathbb{Z}$, and that the ring $\mathbb{Z}[X]/(2X^2 - 3)$ is not integral over $\mathbb{Z}$.
More generally, let A be a factorial ring and $P(X) \in A[X]$ an irreducible polynomial. Show that $A[X]/(P(X))$ is integral over A if and only if the leading coefficient of $P(X)$ is a unit in A (an element in A is called a unit if it is invertible in A).

Exercise 4. Let d be a square-free integer. Let $L = \mathbb{Q}(\alpha)$ be a quadratic field, where $\alpha^2 = d$. Let A be the set of elements in L integral over $\mathbb{Z}$. Show the following:

(a) if $d \equiv 2$ or $d \equiv 3 \bmod 4$, then the set A consists of all elements of the form $a + b\alpha$ with $a, b \in \mathbb{Z}$.
(b) if $d \equiv 1 \bmod 4$, then the set A consists of all elements of the form $\frac{1}{2}(u+v\alpha)$ with u and $v \in \mathbb{Z}$ of the same parity.

Hint: Any element in L is of the form $a+b\alpha$ with $a, b \in \mathbb{Q}$. Compute the minimal polynomial of $a + b\alpha$ over $\mathbb{Q}$, then apply Exercise 2.

Using the explicit description of A from (a) and (b), show that

(c) the set A is a ring.
(d) the set A is a free $\mathbb{Z}$-module of rank 2.